International Journal of Theoretical Physics, Vol. 46, No. 6, June 2007 (© 2007)
DOI: 10.1007/s10773-006-9317-7

Exact Polynomial Solution of $\mathcal{PT}$-/Non-$\mathcal{PT}$-Symmetric and Non-Hermitian Modified Woods–Saxon Potential by the Nikiforov–Uvarov Method

Sameer M. Ikhdair[1] and Ramazan Sever[2]

Received August 21, 2006; accepted November 13, 2006
Published Online: January 18, 2007

Using the Nikiforov–Uvarov (NU) method, the bound state energy eigenvalues and eigenfunctions of the $\mathcal{PT}$-/non-$\mathcal{PT}$-symmetric and non-Hermitian modified Woods–Saxon (WS) model potential with the real and complex-valued energy levels are obtained in terms of the Jacobi polynomials. According to the $\mathcal{PT}$-symmetric quantum mechanics, we exactly solved the time-independent Schrödinger equation with same potential for the s-states and also for any l-state as well. It is shown that the results are in good agreement with the ones obtained before.

KEY WORDS: energy eigenvalues and eigenfunctions; modified WS potential; $\mathcal{PT}$-symmetry; non-Hermitian potential; NU method.

PACS Number(s): 03.65.−w; 02.30.Gp; 03.65.Ge; 68.49.−h; 24.10.Ht; 03.65.Db; 12.39.Pn; 71.15.Dx; 02.30.Fn.

1. INTRODUCTION

A large variety of potentials with the real or complex form are encountered in various fields of physics. A consistent physical theory of quantum mechanics in terms of Hermitian Hamiltonians is built on a complex Hamiltonian that is not Hermitian, but the energy levels are real and positive as a consequence of $\mathcal{PT}$-symmetry (space-time reflection symmetry). By definition, a Hamiltonian is said to be $\mathcal{PT}$ -symmetric when $[\mathcal{PT}, H] = 0$, i.e., $\mathcal{PT}H(\mathcal{PT})^{-1} = (\mathcal{PT})^{-1}H\mathcal{PT} = H$, where $\mathcal{P}$ and $\mathcal{T}$ are, respectively, the operators of parity (or space) and time-reversal (complex conjugation) transformations, i.e., $\mathcal{P}: x \to -x$, $\mathcal{P}: V(x) \to V(-x)$, $\mathcal{P}: p \to -p$, $\mathcal{T}: i \to -i$, $\mathcal{T}: iI \to -iI$ and $\mathcal{PT}: p \to p$, where x, p, and I, are, respectively, the position,

[1] Department of Physics, Near East University, Nicosia, North Cyprus, Mersin 10, Turkey; e-mail: sikhdair@neu.edu.tr

[2] Department of Physics, Middle East Technical University, 06531 Ankara, Turkey; e-mail: sever@metu.edu.tr

momentum and identity operators acting on the Hilbert space $\mathcal{H} = L^2(\mathcal{R})$ and $i = \sqrt{-1}$. Note that this applies only for the system whose classical position x and momentum p are real (Bender and Boettcher, 1998b; Bender *et al*., 1999a,b; Delabaere and Trinh, 2000a; Khare and Mandal, 2000; Mostafazadeh, 2002a,b,c). It is also known that $\mathcal{PT}$-symmetry does not necessarily lead to completely real spectrum, and there are several potentials where part or all of the energy spectrum are complex. The Schrödinger equation (SE) for the real (Hermitian) potentials are investigated to generate the real energy eigenvalues which are of much interest (Khare and Mandal, 2000; Mostafazadeh, 2002a,b,c). Bender *et al*. for the first time and latter others have investigated several complex potentials on the $\mathcal{PT}$-symmetric quantum mechanics. The main reason for the growing recent interest in $\mathcal{PT}$-symmetry (Khare and Mandal, 2000; Mostafazadeh, 2002a,b,c) is that the eigenvalues (spectrum) of every Hamiltonian is real ($\mathcal{PT}$-symmetry is exact) or come in complex conjugate pairs of complex eigenvalues ($\mathcal{PT}$-symmetry is spontaneously broken) (Bender and Boettcher, 1998b; Bender *et al*., 1999a,b, 2001; Cannato *et al*., 1998; Delabaere and Trinh, 2000a,b; Eğrifes *et al*., 1999a,b; Fernández *et al*., 1998; Japaridze, 2002; Khare and Mandal, 2000; Kretschmer and Szymanowski, 2001a; Mezincescu, 2000; Mondal *et al*., 2001; Mostafazadeh, 2002a,b,c; Shin, 2001; Şimşek and Eğrifes, 2004; Znojil and Tater, 2001). Afterwards, non-Hermitian Hamiltonians with real or complex spectra have been studied by using numerical and analytical techniques (Ahmed, 2001a; Bagchi and Quesne, 2000; Berkdemir *et al*., 2005, 2006a; Yeşiltaş *et al*., 2003).

Various different techniques have been employed in solving the above mentioned potential cases. One of these methods which makes it possible to present the theory of special functions by starting from a differential equation has been developed by Nikiforov and Uvarov (NU) method (Nikiforov and Uvarov, 1988). This method is based on solving the time-independent SE by reducing it into a generalized equation of hypergeometric form. Exact solution of SE for central potentials has generated much interest in recent years. So far, these potentials are the parabolic type potential (Barton, 1986), the Eckart potential (Flügge, 1971; Landau and Lifshitz, 1958), the Fermi-step potential (Flügge, 1971; Landau and Lifshitz, 1958), the Rosen-Morse potential (Morse and Feshbach, 1953), the Ginocchio barrier (Sahu *et al*., 2002), the Scarf barriers (Khare and Sukhatme, 1988), the Morse potential (Ahmed, 1991) and a potential which interpolates between Morse and Eckart barriers (Ahmed, 1993). Many authors have studied on exponential type potentials (Barclay *et al*., 2002; Jia *et al*., 2002; Lévai and Znojil, 2002; Yeşiltaş *et al*., 2003; Znojil, 1999) and quasi exactly solvable quadratic potentials (Ahmed, 2001b; Bender and Boettcher, 1998a; Znojil, 2000). In addition, Dirac, Klein–Gordon, and Duffin–Kemmer–Petiau equations for a Coulomb type (generalized Hulthén) potential are solved by using this method (Eğrifes and Sever, 2005; Şimşek and Eğrifes, 2004; Yasuk *et al*., 2005). The exact solutions for these models have been obtained analytically.

So far, we have solved the nonrelativistic and semi-relativistic wave equations using the statistical model (Bekmezci *et al*., 1993; Ikhdair *et al*., 1994a,b), a different approach to the shifted large 1/N expansion technique (Ikhdair and Sever, 1992a,b,c, 1993a, 2003, 2004, 2005a,c,e, 2006a,c; Ikhdair *et al*., 1992, 1993a,b) and also the shifted large $1/\bar{l}$ expansion technique (Ikhdair, 2005) with a wide group of static phenomenological and QCD-motivated potentials to produce the heavy and light quarkonium spectra. In addition, the energy eigenvalues of the bound states of an electron in the general exponential cosine screened Coulomb potential are obtained using the shifted large 1/N expansion method (Ikhdair and Sever, 1993b) and in a novel perturbation method (Ikhdair and Sever, 2005b, 2006b,d). In this work, we solve the SE using NU method (Berkdemir *et al*., 2005, 2006a; Eğrifes *et al*., 1999a,b; Ikhdair and Sever, 2005d; Nikiforov and Uvarov, 1988) with some well-known WS potential (Hagino *et al*., 2001). This potential form has been used widely in analysis of heavy-ion reaction and has enjoyed success (Hagino *et al*., 2001). It is selected for a shell model which can be used for describing metallic clusters in a successful way and for lighting the central part of the interaction neutron with one heavy nucleus (Bulgac and Lewenkopf, 1993; Hamamato *et al*., 2001).

In the present work, the energy eigenvalues and eigenfunctions of the Hermitian and non-Hermitian form of the modified WS potential (generalized WS-plus-the square of generalized WS potential) are calculated by employing the NU method. The aim in this paper is to show that the centrifugal part of SE can be reduced into the generalized WS square potentials.

The contents of this paper are as follows: In Section 2 we breifly present the modified form of WS model potential inspired from the SUSYQM. In Section 3, we present Nikiforov–Uvarov method and also the solution of the Schrödinger equation with Hermitian form of the modified WS model potential for the $l = 0$ and $l \neq 0$ cases. In Section 4, the $\mathcal{PT}-$symmetric and non-$\mathcal{PT}$-symmetric non-Hermitian potential forms are also investigated. Results are given in Section 5. Finally, we give the summary in Section 6.

2. MODIFIED WOODS–SAXON POTENTIAL

The motion of the free electrons which have conclusive influence on the abundance of metallic clusters is a vital problem in the nuclear physics. These electrons are moving in well defined orbitals, around the central nucleus and in a mean field potential which is produced by the positively charged ions and the rest of electrons. In the mean field potential, the details of the potential are described by free parameters such as depth, width and slope of the potential, which have to be fitted to experimental observations. The interactions between nuclei are commonly described by using a potential that consist of the Coulomb and the nuclear potentials. These potentials are usually taken to be of the form of usual

WS model potential (Brack, 1993) which has been widely used in the analysis of heavy-ion reactions (Hagino *et al*., 2001). Recently, Arai (1991) has introduced the deformed hyperbolic potentials using the definitions of the deformed hyper functions. As an example, the inter-nuclear potential, spherically symmetric, is assumed to have the superposition of an attractive generalized WS potential-plus-the repulsive square of a WS potential, i.e., modified WS potential (Berkdemir *et al*., 2006b) in the form:

$$V_q(r;R) = -V_1 \frac{e^{-\left(\frac{r-R}{a}\right)}}{1+qe^{-\left(\frac{r-R}{a}\right)}} + V_2 \left(\frac{e^{-\left(\frac{r-R}{a}\right)}}{1+qe^{-\left(\frac{r-R}{a}\right)}} \right)^2, \quad q \geq 1, \quad R \gg a \quad (1)$$

where r stands for the center-of-mass distance between the projectile and the target nuclei. The parameters of the nuclear potential are given as follows: $R = r_0 A^{1/3}$ is to define the confinement barrier position value of the corresponding spherical nucleus or the width of the potential, A is the target mass number, r_0 is the radius parameter, V_1 controls the barrier height of the Coulombic part, a is the surface diffuseness parameter has to control its slope, is usually adjusted to the experimental values of ionization energies (Hagino *et al*., 2001) and V_2 is an introduced parameter for the second part of Eq. (1) (it transforms like potential barrier) (Berkdemir *et al*., 2006b). Further, q is a deformation parameter, the strength of the exponential part other than unity, set to determine the shape of potential and is arbitrarily taken to be a real constant within the potential. Hence, its worthwhile to note, the proposed potential form in (1) is therefore consisting of the generalized WS-plus-the square of the generalized WS potential. Further, we remark that the spatial coordinates in the potential are not deformed and thus the potential still remains spherical. We have to note also that, for some specific q values this potential reduces to the well-known types, such as for $q = 0$ to the exponential potential and for $q = -1$ and $a = \delta^{-1}$ to the modified Hulthén potential (Ikhdair and Sever, 2005d).

To study any quantum physical system characterized by the empirical potential given in Eq. (1), we solve the original SE:

$$\left(\frac{p^2}{2m} + V(r) \right) \chi(\mathbf{r}) = E\chi(\mathbf{r}), \quad (2)$$

where the classical phase space is assumed to be real, i.e., r and p are the standard Hermitian operators representing the position and momentum of a particle of mass m. Employing the separation of variables

$$\chi(\mathbf{r}) = \frac{1}{r}\psi(r)Y(\theta,\phi), \quad (3)$$

leads to the simple radial SE, for all angular momentum states, of the type

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(r)}{dr^2}+\left(V(r)+\frac{l(l+1)\hbar^2}{2mr^2}\right)\psi(r)=E_n\psi(r). \tag{4}$$

For this, our aim is to solve the last equation for the given modified WS model potential given in (1) for the cases $l=0$ and $l\neq 0$ using the NU method which will be introduced breifly in Section 3 and employed latter on in the coming sections.

3. THE NIKIFOROV–UVAROV METHOD

The Nikiforov–Uvarov (NU) method provides us an exact solution of Eq. (4) for certain kind of potentials among them the one given in Eq. (1) (Nikiforov and Uvarov, 1988). This method is based upon the solutions of general second order linear differential equation with special orthogonal functions (Arai, 1991). For a given real or complex potentials, the $\mathcal{PT}$-symmetric one-dimensional ($1D$) SE is reduced to a generalized equation of hypergeometric type with an appropriate $s=s(x)$ coordinate transformation. Thus, it takes the form:

$$\psi_n''(s)+\frac{\widetilde{\tau}(s)}{\sigma(s)}\psi_n'(s)+\frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s)=0, \tag{5}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are polynomials, at most of second-degree, and $\widetilde{\tau}(s)$ is of a first-degree polynomial. To find a particular solution for Eq. (5) by separation of variables, we use the transformation given by

$$\psi_n(s)=\phi_n(s)y_n(s). \tag{6}$$

This reduces SE, Eq. (5), into an equation of hypergeometric type:

$$\sigma(s)y_n''(s)+\tau(s)y_n'(s)+\lambda y_n(s)=0, \tag{7}$$

where $\phi(s)$ is found to satisfy the condition $\phi'(s)/\phi(s)=\pi(s)/\sigma(s)$. Further, $y_n(s)$ is the hypergeometric type function whose polynomial solutions are given by Rodrigues relation

$$y_n(s)=\frac{B_n}{\rho(s)}\frac{d^n}{ds^n}[\sigma^n(s)\rho(s)], \tag{8}$$

where B_n is a normalization constant and the weight function $\rho(s)$ must satisfy the condition (Nikiforov and Uvarov, 1988)

$$\frac{w'(s)}{w(s)}=\frac{\tau(s)}{\sigma(s)}, \tag{9}$$

where $w(s) = \sigma(s)\rho(s)$. The function $\pi(s)$ and the eigenvalue parameter λ required for this method are defined as

$$\pi(s) = \frac{\sigma'(s) - \widetilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \widetilde{\tau}(s)}{2}\right)^2 - \widetilde{\sigma}(s) + k\sigma(s)}, \tag{10}$$

and

$$\lambda = k + \pi'(s). \tag{11}$$

Here, $\pi(s)$ is a polynomial with the parameter s and the determination of k is the essential point in the calculation of $\pi(s)$. Thus, for the determination of k, the discriminant under the square root is being set equal to zero and the resulting second-order polynomial has to be solved for its roots $k_{+,-}$. Hence, a new eigenvalue equation for the SE becomes

$$\lambda_n + n\tau'(s) + \frac{n\,(n-1)}{2}\sigma''(s) = 0, \quad (n = 0, 1, 2, \ldots) \tag{12}$$

where

$$\tau(s) = \widetilde{\tau}(s) + 2\pi(s), \tag{13}$$

and it must have a negative derivative. Let us now start solving SE for the $\mathcal{PT}$ modified WS using the NU technique for $l = 0$ and $l \neq 0$ cases as follows.

3.1. Solution for the $l = 0$ Case

Let us start our study by rewriting the potential in Eq. (1) in a 1D form

$$V_q(x) = -V_1 \frac{e^{-\alpha x}}{1 + qe^{-\alpha x}} + V_2 \frac{e^{-2\alpha x}}{(1 + qe^{-\alpha x})^2}, \quad q \geq 1, \tag{14}$$

where some assignment of $1D$ parameter $r - R = x$, for which $x \in (-\infty, \infty)$ and $\alpha = 1/a$ are done. We proceed to calculate the energy eigenvalues and their eigenfunctions by substituting the Hermitian real-valued potential given by Eq. (14) into the $1D$ $\mathcal{PT}$-symmetrical Hermitian SE with zero angular momentum states,

$$\psi''_{nq}(x) + \frac{2m}{\hbar^2}\left[E_n + \frac{V_1 e^{-\alpha x}}{1 + qe^{-\alpha x}} - \frac{V_2 e^{-2\alpha x}}{(1 + qe^{-\alpha x})^2}\right]\psi_{nq}(x) = 0, \tag{15}$$

and further employing the following convenient transformation, $s(x) = (1 + qe^{-\alpha x})^{-1}$ to transform Eq. (15) into the form of hypergeometric type equation

given in Eq. (5) as

$$\frac{d^2\psi_{nq}(s)}{ds^2}+\frac{1-2s}{s-s^2}\frac{d\psi_{nq}(s)}{ds} + \frac{\frac{2ma^2}{\hbar^2}[-\widetilde{V}_2 s^2+(2\widetilde{V}_2-\widetilde{V}_1)s+E_n+\widetilde{V}_1-\widetilde{V}_2]}{(s-s^2)^2}\psi_{nq}(s)=0, \tag{16}$$

where $\widetilde{V}_1 = V_1/q$ and $\widetilde{V}_2 = V_2/q^2$. Furthermore, introducing the following dimensionless abbreviations:

$$\epsilon^2 = -\frac{2ma^2E_n}{\hbar^2} > 0\,(E_n<0), \quad \beta^2 = \frac{2ma^2\widetilde{V}_1}{\hbar^2}(\beta^2>0),$$
$$\gamma^2 = \frac{2ma^2\widetilde{V}_2}{\hbar^2}\,(\gamma^2>0), \tag{17}$$

finally gives the following simple hypergeometric type equation which is given by Eq. (5):

$$\psi''_{nq}(s)+\frac{1-2s}{s-s^2}\psi'_{nq}(s)+\frac{[-\gamma^2s^2+(2\gamma^2-\beta^2)s+\beta^2-\gamma^2-\epsilon^2]}{(s-s^2)^2}\psi_{nq}(s)=0, \tag{18}$$

where the wave functions $\psi_n(s)$ in the last equation satisfies the following boundary conditions:

$$\psi_{nq}(s)=\begin{cases}0, & \text{at } s=1\ (r\to\infty),\\ 0, & \text{at } s\to 0\ \ (r=0)\ \text{ and }\ R\gg a.\end{cases} \tag{19}$$

After comparing Eq. (18) with Eq. (5), we obtain the following associated polynomials:

$$\widetilde{\tau}(s)=1-2s, \quad \sigma(s)=s-s^2,$$
$$\widetilde{\sigma}(s)=-\gamma^2s^2+(2\gamma^2-\beta^2)s+\beta^2-\gamma^2-\epsilon^2. \tag{20}$$

Substituting these polynomials into Eq. (10), with $\sigma'(s)=1-2s$, we achieve the linear function

$$\pi(s)=\pm\sqrt{(\gamma^2-k)s^2+(\beta^2-2\gamma^2+k)s+\epsilon^2+\gamma^2-\beta^2}. \tag{21}$$

Further, the discriminant of the upper expression under the square root has to be set equal to zero:

$$\Delta = k^2+(4\epsilon^2-2\beta^2)k+\beta^4-4\gamma^2\epsilon^2=0, \tag{22}$$

giving the double roots, i.e., $k_{+,-}=\beta^2-2\epsilon^2\pm 2\varsigma\epsilon$, where $\varsigma=\sqrt{\epsilon^2+\gamma^2-\beta^2}$. Afterwards, these two roots are substituted into Eq. (21) giving two possible

functions for each k :

$$\pi(s) = \pm \begin{cases} (\varsigma - \epsilon)\, s - \varsigma; & \text{for } k_+ = \beta^2 - 2\epsilon^2 + 2\varsigma\epsilon, \\ (\varsigma + \epsilon)\, s - \varsigma; & \text{for } k_- = \beta^2 - 2\epsilon^2 - 2\varsigma\epsilon. \end{cases} \tag{23}$$

Therefore, the polynomial $\tau(s)$ in Eq. (13) whose derivative has a negative value is found by a suitable choice of the polynomial $\pi(s)$ for $k_- = \beta^2 - 2\epsilon^2 - 2\varsigma\epsilon = \gamma^2 - (\varsigma + \epsilon)^2$ from Eq. (23):

$$\pi(s) = -(\varsigma + \epsilon)\, s + \varsigma, \tag{24}$$

$$\begin{aligned} \tau(s) &= -2(1 + \varsigma + \epsilon)\, s + (1 + 2\varsigma), \quad \tau'(s) = -2(1 + \varsigma + \epsilon) \\ &= -1 - \sqrt{1 + 4\gamma^2} + 2n, \end{aligned} \tag{25}$$

where the radial quantum number n is a non-negative integer describing the quantization of the bound-states. Therefore, from Eqs. (11) and (12), we find a new eigenvalue equations as

$$\lambda = \gamma^2 - (\varsigma + \epsilon)(\varsigma + \epsilon + 1), \tag{26}$$

and

$$\lambda_n = n(n+1) + 2n(\varsigma + \epsilon). \tag{27}$$

Afterwards, setting $\lambda_n = \lambda$ and then solving for ϵ, we find:

$$\begin{aligned} \epsilon_n^2(a, q) = \frac{1}{16}[&((2n+1) - \sqrt{1 + 4\gamma^2}) \\ &+ 4(\beta^2 - \gamma^2)((2n+1) - \sqrt{1 + 4\gamma^2})^{-1}]^2. \end{aligned} \tag{28}$$

Therefore, substituting the values of ϵ^2, β^2 and γ^2 into Eq. (28), we can immediately obtain the exact energy spectrum for the $\mathcal{PT}$-symmetric modified WS potential given by (1) as

$$\begin{aligned} E_{nq} = -\frac{\hbar^2}{2ma^2} \Bigg[&\frac{1}{16} \left((2n+1) - \sqrt{1 + \frac{8ma^2 V_2}{\hbar^2 q^2}} \right)^2 + \frac{ma^2}{\hbar^2 q^2} (qV_1 - V_2) \\ &+ \left(\frac{2ma^2}{\hbar^2 q^2} \right)^2 (qV_1 - V_2)^2 \left((2n+1) - \sqrt{1 + \frac{8ma^2 V_2}{\hbar^2 q^2}} \right)^{-2} \Bigg], \quad V_2 \neq 0. \end{aligned} \tag{29}$$

The last equation is found to be consistent with the analytical solutions of the energy eigenvalues for the modified WS potential obtained by the SUSYQM approach in Berkdemir *et al*. (2006b). Further, when the parameter q is taken to

be one and the atomic units (au) ($\hbar = m = c = e = 1$) are used, Eq. (29) can be reduced into the following simple form:

$$E_n = -\frac{1}{2}\left[\left(\frac{(2n+1)-\sqrt{1+8a^2V_2}}{4a}\right)^2 + 4a^2(V_1 - V_2)^2((2n+1)-\sqrt{1+8a^2V_2})^{-2} + V_1 - V_2\right]. \quad (30)$$

where $V_2 = V_1^2$. The above equation indicates that we deal with a family of modified WS spherical potentials if appropriate changes in the parameters a and V_1 are done. In addition, if the parameter V_2 in Eq. (15) is adjusted to zero for s-state (i.e., the generalized WS model potential, cf. Berkdemir *et al.*, 2006a), the Schrödinger equation, in the present case, must be solved once again for its energy eigenstates without regarding of the centrifugal barrier potential and the deformation parameter q.[3]

Let us now find the corresponding wavefunctions. As stated in Eq. (6), in the NU method, the wave function is constructed as a combination of two independent parts. The function $y_{nq}(s)$ is the polynomial solution of the hypergeometric-type equation which is given by Eq. (9) and described with the following weight function:

$$\rho_{nq}(s) = s^{2\varsigma}(1-s)^{2\epsilon}; \quad -2\varsigma = \left[(2n+1) - \sqrt{1+\frac{8ma^2V_2}{\hbar^2q^2}}\right] + 2\epsilon. \quad (31)$$

Substituting Eq. (30) into the Rodrigues relation given by Eq. (8), the polynomial $y_{nq}(s)$ is calculated as

$$y_{nq}(s) = C_{nq}s^{-2\varsigma}(1-s)^{-2\epsilon}\frac{d^n}{ds^n}[s^{n+2\varsigma}(1-s)^{n+2\epsilon}], \quad (32)$$

where C_{nq} stands for the normalization constant. The functions $y_{nq}(s)$ are, up to a numerical factor, in the form of Jacobi Polynomials which is one of the classical orthogonal polynomials, i.e., $y_{nq}(s) \simeq P_n^{(2\varsigma,2\epsilon)}(1-2s)$; which is valid physically in the interval $(0 \le r < \infty \to 0 \le s \le 1)$ (Sezgo, 1939). On the other hand, the other part of the wave function can be found by substituting $\pi(s)$ and $\sigma(s)$ into the expression $\phi'(s)/\phi(s) = \pi(s)/\sigma(s)$ and then solving the resulting differential equation to obtain

$$\phi_{nq}(s) = s^{\varsigma}(1-s)^{\epsilon}. \quad (33)$$

[3] The exact bound s-states energy eigenvalues, in au, becomes $E_n(a=1, q \to 1) = -\frac{1}{8}[n+1+\frac{2V_1}{(n+1)}]^2$, which is consistent with previous works on SUSYQM (cf. formula (40) in Berkdemir, Berkdemir, and Sever, (2006b)).

Finally, combining the Jacobi polynomials and $\phi_{nq}(s)$ in Eq. (33), the s-wave functions could be determined as

$$\psi_{nq}(s) = N_{nq} s^{\varsigma}(1-s)^{\epsilon} P_n^{(2\varsigma,2\epsilon)}(1-2s),$$
$$\epsilon = \sqrt{-2ma^2E_n/\hbar^2}, \qquad s = (1+qe^{-x/a})^{-1}, \tag{34}$$

where N_{nq} is a new normalization constant. The last equation satisfies the requirements; $\psi_n(s) = 0$ at $s = 1$ $(r \to \infty)$ and $\psi_n(s) = 0$ at $s = (1+e^{-x/a})^{-1} \approx 0$ $(r = 0)$, which is valid for realistic nuclei because the confinement barrier radius R is much bigger than the surface diffuseness parameter, i.e., $R \gg a$. Therefore, the wave function, $\psi_{nq}(s)$ in Eq. (34) is valid physically in the closed interval shown above, i.e., $s \in [0, 1]$ or $r \in [0, \infty)$. In addition, the wave functions satisfy the normalization condition

$$\int_0^\infty |\psi_{nq}(x)|^2 dx = 1 = \int_0^1 |\psi_{nq}(s)|^2 ds, \tag{35}$$

where N_{nq} can be determined via

$$1 = \int_0^1 |\psi_{nq}(x)|^2 dx = N_{nq}^2 \int_0^1 s^{2\varsigma}(1-s)^{2\epsilon}\left[P_n^{(2\varsigma,2\epsilon)}(1-2s)\right]^2 ds. \tag{36}$$

We now make use of the fact that the Jacobi polynomials, $P_n^{(\rho,\nu)}(\xi)$, can be explicitly written in two different ways (Magnus *et al.*, 1966):

$$P_n^{(\rho,\nu)}(\xi) = 2^{-n}\sum_{p=0}^{n}(-1)^{n-p}\binom{n+\rho}{p}\binom{n+\nu}{n-p}(1-\xi)^{n-p}(1+\xi)^p, \tag{37}$$

$$P_n^{(\rho,\nu)}(\xi) = \frac{\Gamma(n+\rho+1)}{n!\Gamma(n+\rho+\nu+1)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(n+\rho+\nu+r+1)}{\Gamma(r+\rho+1)}\left(\frac{\xi-1}{2}\right)^r, \tag{38}$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$. Using Eqs. (37)–(38), we obtain the explicit expressions for $P_n^{(2\varsigma,2\epsilon)}(1-2s)$:

$$P_n^{(2\varsigma,2\epsilon)}(1-2s) = (-1)^n\Gamma(n+2\varsigma+1)\Gamma(n+2\epsilon+1)$$
$$\times\sum_{p=0}^{n}\frac{(-1)^p q^{n-p}}{p!(n-p)!\Gamma(p+2\epsilon+1)\Gamma(n+2\varsigma-p+1)}s^{n-p}(1-qs)^p, \tag{39}$$

$$P_n^{(2\varsigma,2\epsilon)}(1-2s) = \frac{\Gamma(n+2\varsigma+1)}{\Gamma(n+2\varsigma+2\epsilon+1)}\sum_{r=0}^{n}\frac{(-1)^r q^r\Gamma(n+2\varsigma+2\epsilon+r+1)}{r!(n-r)!\Gamma(2\varsigma+r+1)}s^r. \tag{40}$$

Substituting Eqs. (39) and (40) into Eq. (36), one obtains

$$1 = N_{nq}^2(-1)^n \frac{\Gamma(n+2\epsilon+1)\Gamma(n+2\varsigma+1)^2}{\Gamma(n+2\varsigma+2\epsilon+1)} \sum_{p,r=0}^{n} \times \frac{(-1)^{p+r} q^{n+r-p} \Gamma(n+2\varsigma+2\epsilon+r+1)}{p!r!(n-p)!(n-r)!\Gamma(p+2\epsilon+1)\Gamma(n+2\varsigma-p+1)\Gamma(2\varsigma+r+1)} I_{nq}(p,r), \tag{41}$$

where

$$I_{nq}(p,r) = \int_0^1 s^{n+2\varsigma+r-p}(1-qs)^{p+2\epsilon} ds. \tag{42}$$

Using the following integral representation of the hypergeometric function (Macrobert, 1962; Prudrinkov *et al*., 1986)

$$ {}_2F_1(\alpha_0, \beta_0 : \gamma_0; q) \frac{\Gamma(\alpha_0)\Gamma(\gamma_0-\alpha_0)}{\Gamma(\gamma_0)} = \int_0^1 s^{\alpha_0-1}(1-s)^{\gamma_0-\alpha_0-1}(1-qs)^{-\beta_0} ds, $$
$$ [\mathrm{Re}(\gamma_0) > \mathrm{Re}(\alpha_0) > 0, \quad |\arg(1-q)| < \pi] \tag{43}$$

which gives

$$ {}_2F_1(\alpha_0, \beta_0 : \alpha_0+1; q)/\alpha_0 = \int_0^1 s^{\alpha_0-1}(1-qs)^{-\beta_0} ds, \tag{44}$$

where

$$ {}_2F_1(\alpha_0, \beta_0 : \gamma_0; q) = \frac{\Gamma(\gamma_0)\Gamma(\gamma_0-\alpha_0-\beta_0)}{\Gamma(\gamma_0-\alpha_0)\Gamma(\gamma_0-\beta_0)}, $$
$$ (\mathrm{Re}(\gamma_0-\alpha_0-\beta_0) > 0, \quad \mathrm{Re}(\gamma_0) > \mathrm{Re}(\beta_0) > 0). \tag{45}$$

For the present case, with the help of Eq. (45), when $\alpha_0 = n+2\varsigma+r-p+1$, $\beta_0 = -p-2\epsilon$, and $\gamma_0 = \alpha_0+1$ are substituted into Eq. (44), we obtain

$$I_{nq}(p,r) = \frac{{}_2F_1(\alpha_0, \beta_0 : \gamma_0; q)}{\alpha_0} = \frac{(n+2\varsigma+r-p+1)!(p+2\epsilon)!}{(n+2\varsigma+r-p+1)(n+2\varsigma+r+2\epsilon+2)!}. \tag{46}$$

Now we shall study the case where $V_2 = 0$, so we need to solve the $1D$ Schrödinger equation for the generalized WS potential given by Eq. (14). For this, after applying the previous transformation into Eq. (15), it gives

$$\psi''_{nq}(s) + \frac{1-2s}{s-s^2}\psi'_{nq}(s) + \frac{[-\beta^2 s + \beta^2 - \epsilon^2]}{(s-s^2)^2}\psi_{nq}(s) = 0, \tag{47}$$

for which.

$$\widetilde{\tau}(s) = 1 - 2s, \quad \sigma(s) = s - s^2,$$
$$\widetilde{\sigma}(s) = -\beta^2 s + \beta^2 - \epsilon^2, \tag{48}$$
$$\epsilon^2 = -\frac{2ma^2 E_n}{\hbar^2} > 0 \quad (E_n < 0), \quad \beta^2 = \frac{2ma^2 V_1}{q\hbar^2} \quad (\beta^2 > 0).$$

Following a procedure analogous to the previous case we see that this time when $\pi(s) = -(c+\epsilon)s + c$ is chosen for $k_- = -(c+\epsilon)^2$, $c = \sqrt{\epsilon^2 - \beta^2}$,

$$\tau(z) = -2(1 + c + \epsilon)s + (1 + 2c), \tag{49}$$

could be obtained. Further, the eigenvalues are

$$\lambda_n = n^2 + n + 2n(c+\epsilon), \quad \lambda = -(c+\epsilon)(c+\epsilon+1), \tag{50}$$

giving the exact energy spectrum of the generalized WS potential as

$$E_{nq} = -\frac{\hbar^2}{2ma^2}\left[\frac{(n+1)}{2} + \frac{ma^2 V_1}{(n+1)\hbar^2 q}\right]^2, \quad 0 \le n < \infty. \tag{51}$$

which are consistent with the work in Berkdemir, Berkdemir, and Sever (2006b) if one lets $V_2 = 0$ in Eq. (14). As an illustrative example. We consider the usual Hulthén potential (Ikhdair and Sever, 2005d), i.e., $V(r) = -Ze^2\delta\frac{\exp(-\delta r)}{1-\exp(-\delta r)}$. Therefore, for $q = -1$, $V_1 = Ze^2\delta$, and $a = 1/\delta$, Eq. (51) gives the following exact s-wave bound-states spectra

$$E_n = -\frac{me^4 z^2}{2\hbar^2}\left[\frac{1}{n+1} - \frac{(n+1)}{2}\eta\right]^2, \quad 0 \le n < \infty, \quad \eta = \frac{\hbar^2\delta}{me^2 z}, \tag{52}$$

which, consequently, yields the critical range parameter $\delta_c = \frac{2}{(n+1)^2}$ and from which the bound-states are subject to the condition $n < \sqrt{2/\delta} - 1$. Hence, we remark that the above result was found in recent works on Hulthén potential (cf. (Aktaş and Sever, 2004; Filho and Ricotta, 1995; Ikhdair and Sever, 2005d)).

On the other hand, the radial wave function in the current case becomes

$$\psi_{nq}(s) = D_{nq} s^c (1-s)^\epsilon P_n^{(2c,2\epsilon)}(1-2s), \tag{53}$$

with $s = (1 + qe^{-x/a})^{-1}$ and D_{nq} is a new normalization constant.

3.2. Solution for the $l \neq 0$ Case

We present the Hamiltonian for the modified WS potential for the $l \neq 0$ case as

$$H = \frac{p^2}{2m} - V_1\frac{e^{-\left(\frac{r-R}{a}\right)}}{1 + qe^{-\left(\frac{r-R}{a}\right)}} + \frac{l(l+1)\hbar^2}{2mr^2}. \tag{54}$$

In order to evaluate the energy spectrum and their eigenfunctions, we introduce the proposed modified WS potential of the form:

$$V_{eff}\ (r;R) = -V_1 \frac{e^{-\left(\frac{r-R}{a}\right)}}{1+qe^{-\left(\frac{r-R}{a}\right)}} + \frac{l(l+1)\hbar^2}{2ma^2}\left(\frac{e^{-\left(\frac{r-R}{a}\right)}}{1+qe^{-\left(\frac{r-R}{a}\right)}}\right)^2, \tag{55}$$

where the second term in Eq. (55) appears to be like the potential barrier term of Eq. (54). Thus, comparing Eq. (55) with its counterpart Eq. (1), we can make the convenient transformation replacement of $V_2 \to \frac{\hbar^2 l(l+1)}{2ma^2}$ which provides

$$V_{eff}\ (r;R) = -V_1 \frac{1}{e^{\left(\frac{r-R}{a}\right)}+q} + \frac{l(l+1)\hbar^2}{2ma^2\left(e^{\left(\frac{r-R}{a}\right)}+q\right)^2}, \tag{56}$$

where $a = R/(q+1)$. The expression for the lowest energy levels of the potential in Eq. (55), after setting $q = 1$, in au becomes

$$E_{n,\ell} = -\frac{1}{2a^2}\left[\frac{1}{4}((2n+1) - \sqrt{1+4\ell(\ell+1)}) \right.$$
$$\left. + (2a^2V_1 - \ell(\ell+1))((2n+1) - \sqrt{1+4\ell(\ell+1)})^{-1}\right]^2. \tag{57}$$

In this regard, for scattering processes, it has been well accepted that the surface diffuseness parameter a is around $0.63\,fm$ (Chamon *et al.*, 1996; Christensen and Winther, 1976; Hagino *et al.*, 2001; Silva *et al.*, 2001). Much larger diffuseness parameter, ranging between 0.8 and $1.4\,fm$ is needed in order to fit the data (Hagino *et al.*, 2001; Jaminon *et al.*, 1986).

4. NON-HERMITIAN POTENTIAL FORMS

Under $\mathcal{P}$, the spatial coordinates (x, y, z) are replaced by $(-x, -y, -z)$ but r is replaced by r and not $-r$, in the radial wave equation (4). Thus, the s-wave differential equation is not $\mathcal{PT}$-symmetric. The radial Schrödinger wave equation becomes a different differential equation under the action of the $\mathcal{PT}$-operator and does not go into itself. This means that we must solve the problem in $1D$, on the full plane, say x-direction and not in the radial direction r.

4.1. $\mathcal{PT}$-Symmetric and Non-Hermitian Modified WS Case

A potential is said to be a $\mathcal{PT}$-symmetric when it satisfies the $\mathcal{PT}$-symmetry condition for a given $1D$ potential $V(x)$ which is $[V(-x)]^* = V(x)$. Further the Hamiltonian is said to be $\mathcal{PT}$-symmetric when $[\mathcal{PT}\ ,H] = 0$.

We are, now, going to consider a different form of the modified WS potential, at least one of the potential parameters is complex. If $\alpha = 1/a$ is pure complex

(imaginary) parameter, i.e., $\alpha \to i a_I$ whereas V_1, V_2, and q are real, such a potential is written as a complex function

$$V(x) = -V_1 \left(\frac{q + \cos\alpha_I x - i \sin\alpha_I x}{1 + q^2 + 2q\cos\alpha_I x} \right) + V_2 \left(\frac{q + \cos\alpha_I x - i \sin\alpha_I x}{1 + q^2 + 2q\cos\alpha_I x} \right)^2. \tag{58}$$

which possesses $\mathcal{PT}$-symmetric but non-Hermitian. Hence, under joint action of spatial and time reversal, we obtained $[\mathcal{PT}, V(x)] = 0$, i.e., $\mathcal{PT}\, V(x) = V(x)\mathcal{PT}$, for the complex version of the potential function given by Eq. (14). By substituting the above potential into Eq. (15) and repeating similar procedures in obtaining Eq. (29), one can easily obtain the energy eigenvalues as

$$E_{nq} = \frac{\hbar^2}{2m} \left[\frac{\alpha_I \left((2n+1) - \sqrt{1 - \frac{8mV_2}{\hbar^2 q^2 \alpha_I^2}} \right)}{4} \right.$$

$$\left. - \frac{2m}{q^2 \hbar^2 \alpha_I} (qV_1 - V_2) \left((2n+1) - \sqrt{1 - \frac{8mV_2}{\hbar^2 q^2 \alpha_I^2}} \right)^{-1} \right]^2, \quad V_2 \neq 0. \tag{59}$$

Therefore if $8mV_2 < \hbar^2 q^2 \alpha_I^2$, then there exists bound states, otherwise there are no bound states. We also have

$$\lambda = -\frac{2mV_2}{q^2 \alpha_I^2 \hbar^2} + (\varsigma + \epsilon)(\varsigma + \epsilon - i), \tag{60}$$

$$\lambda_n = n(n+1) + 2in\varsigma. \tag{61}$$

Here α_I is an arbitrary real parameter and $i = \sqrt{-1}$. Thus, by choosing the parameter α as purely imaginary, we find the energy eigenvalues obtained for $\mathcal{PT}$-symmetric and non-hermitian modified WS potential are not similar to Eq. (29). A positive energy spectra is obtained if and only if

$$n < \sqrt{\frac{2m}{\hbar^2 \alpha_I^2} \left(\frac{V_1}{q} - \frac{V_2}{q^2} \right)} - \frac{1}{2} \left(1 - \sqrt{1 - \frac{8mV_2}{\hbar^2 \alpha_I^2 q^2}} \right),$$

since the energy eigenvalues of modified WS potential are negative.[4] It appears that there exist bound-states at the eigenvalues are always positive real for $V_2 = 0$, that is,

$$E_n(\alpha_I, q) = \frac{\hbar^2}{2m} \left[\frac{1+n}{2} \alpha_I - \frac{mV_1}{(1+n)\hbar^2 \alpha_I q} \right]^2,$$

[4] Once we set $V_2 = 0$, then $n < \sqrt{\frac{2mV_1}{\hbar^2 \alpha_I^2 q}} - 1$.

otherwise there are no bound-states but can be complex for $V_2 > \frac{\hbar^2\alpha_I^2 q^2}{8m}$. Since these 1D non-Hermitian Hamiltonians were invariant under $\mathcal{PT}$-transformation, they possessed real spectra. Thus, their real spectral properties were linked with their $\mathcal{PT}$-symmetry.

On the other hand, to avoid repeatition, we obtain an eigenfunction for the non-Hermitian potential, Eq. (58), as:

$$\begin{aligned}\psi_{nq}(s) &= N_{nq}s^{i\varsigma}(1-s)^{i\epsilon}P_n^{(2i\varsigma,2i\epsilon)}(1-2s),\\ i\varsigma &= -\frac{1}{2}\left[(2n+1)-\sqrt{1-8mV_2/\hbar^2\alpha_I^2}\right]-i\epsilon, \qquad \mathrm{i}\epsilon = \sqrt{2mE_n/\hbar^2\alpha_I^2},\\ s &= (1+qe^{-i\alpha_I x})^{-1}.\end{aligned} \tag{62}$$

4.2. Non-$\mathcal{PT}$-Symmetric and Non-Hermitian Modified WS Case

It is interesting to note that when the two parameters V_1 and α are imaginary at the same time, i.e., $V_1 \to iV_{1I}$, and $\alpha \to i\alpha_I$, whereas V_2 remains a pure real, the potential in (14) transforms into the form

$$V(x) = -V_{1I}\left(\frac{\sin\alpha_I x + i(q+\cos\alpha_I x)}{1+q^2+2q\cos\alpha_I x}\right) + V_2\left(\frac{q+\cos\alpha_I x - i\sin\alpha_I x}{1+q^2+2q\cos\alpha_I x}\right)^2. \tag{63}$$

The Hamiltonian, in this case, is non-Hermitian and non-$\mathcal{PT}$-symmetric having real spectra. Making substitution of this potential into Eq. (15) and repeating similar procedures in obtaining Eq. (29), one can easily get the energy eigenvalues as

$$\begin{aligned}E_{nq} = \frac{\hbar^2}{2m}\Bigg[&\frac{\alpha_I\left((2n+1)-\sqrt{1-\frac{8mV_2}{\hbar^2q^2\alpha_I^2}}\right)}{4}\\ &-\frac{2m}{\hbar^2q^2\alpha_I}(iqV_{1I}-V_2)\left((2n+1)-\sqrt{1-\frac{8mV_2}{\hbar^2q^2\alpha_I^2}}\right)^{-1}\Bigg]^2, \quad V_2 \neq 0\end{aligned} \tag{64}$$

and

$$\lambda = -\frac{2mV_2}{q^2\alpha_I^2\hbar^2} + (b+\epsilon)(b+\epsilon-i), \tag{65}$$

$$\lambda_n = n(n+1) + 2in(b+\epsilon), \tag{66}$$

where $b = \sqrt{\epsilon^2 + \gamma^2 - i\beta^2}$. The last case has real plus imaginary energy spectra. When we consider the real part of energy eigenvalues an acceptable result is obtained when

$$n < \sqrt{\frac{2m}{\hbar^2\alpha_I^2}\left(\frac{iV_{1I}}{q} - \frac{V_2}{q^2}\right)} - \frac{1}{2}\left(1 - \sqrt{1 - \frac{8mV_2}{\hbar^2\alpha_I^2 q^2}}\right)$$

condition. However, the energy spectrum is not seen at the imaginary part of energy eigenvalues, since it is independent of n.[5]

On the other hand, to avoid repeatition, we obtain an eigenfunction for the non-Hermitian potential as:

$$\psi_{nq}(s) = N_{nq} s^{ib}(1-s)^{i\epsilon} P_n^{(2ib,2i\epsilon)}(1-2s),$$
$$ib = -\frac{1}{2}\left[(2n+1) - \sqrt{1 - 8mV_2/\hbar^2\alpha_I^2}\right] - i\epsilon, \quad \mathrm{i}\epsilon = \sqrt{2ma^2E_n/\hbar^2},$$
$$s = \left(1 + qe^{-i\alpha_I(r-R)}\right)^{-1}. \tag{67}$$

5. RESULTS

As a first attempt, we compute here the binding energy spectrum from Eq. (30) for the proposed modified WS potential. All we need simply is to obtain an optimized set of parameters for V_1 and a matching each confinement radius value R which describe well the physical system in question. Consequently, we choose the sets of parameters already been fitted for the usual WS potential by utilizing the optimization procedures (Costa *et al*., 1999).

Therefore, for $V_2 = 0$, by means of Eq. (51) (cf. Berkdemir *et al*., 2006a), we calculate the ns-states binding energy spectrum in units of MeV for five confinement radius values (i.e., $R = 0.5, 1.0, 2.0, 3.0, 4.0$) in Table I. Further, we produce the bound energy eigenvalues of the lowest $1p$- and $1d$-states for the above five confinement radius values in Table II by means of Eq. (57). As we have used throughout the au, our energy numerical calculations are made in units of $2RY = 27.212\,\mathrm{eV}$ (Taylor, 1984) and so that distances are measured in the Bohr radius a_0.

On the other hand, it is illustrated in Fig. 1, the general form of the generalized WS potential, $V_2 = 0$ or s-states, for various values of the deformation parameter $q = 1, 3$ and 7. This is made by utilizing the set of fitting parameters of $R = 3$ case already found by Costa *et al*. (1999) (cf. Table I). Moreover, the same set of parameters together with $V_2 = V_1^2$ (p-states) are used in drawing the general form of the modified WS potential in Fig. 2. The s-states energy shapes, in au, of the

[5] Once we set $V_2 = 0$, then $n < \sqrt{\frac{i2mV_{1I}}{\hbar^2\alpha_I^2 q}} - 1$ which agrees with Berkdemir, Berkdemir, and Sever, (2005, 2006).

Table I. Calculated energy eigenvalues of the usual WS potential for the s-states in MeV

State (n)	R	$1/a$	V_1	E_n
0	0.5	125.4	4057.7	−122.94
0	1.0	23.1	247.3	−6.74
0	2.0	5.8	82.1	−3.96
0	3.0	3.5	123.8	−18.75
0	4.0	2.8	318.4	−180.3
1	0.5			−272.73
1	1.0			−11.02
1	2.0			−2.26
1	3.0			−6.11
1	4.0			−48.42
2	0.5			−538.19
2	1.0			−19.87
2	2.0			−2.45
2	3.0			−3.95
2	4.0			−24.12
3	0.5			−911.93
3	1.0			−32.50
3	2.0			−3.12
3	3.0			−3.42
3	4.0			−15.75
4	0.5			−1393.01
4	1.0			−48.80
4	2.0			−4.09
4	3.0			−3.41
4	4.0			−12.04

Table II. Calculated bound energy eigenvalues for lowest $1p$-and $1d$-states in MeV

State (n)	ℓ	R	$1/a$	V_1	$E_{n,\ell}$
0	1	0.5	125.4	4057.7	−12.53
0	1	1.0	23.1	247.3	−0.01
0	1	2.0	5.8	82.1	−1.72
0	1	3.0	3.5	123.8	−15.38
0	1	4.0	2.8	318.4	−163.18
0	2	0.5	125.4	4057.7	−29.45
0	2	1.0	23.1	247.3	−0.52
0	2	2.0	5.8	82.1	−0.24
0	2	3.0	3.5	123.8	−3.46
0	2	4.0	2.8	318.4	−41.85

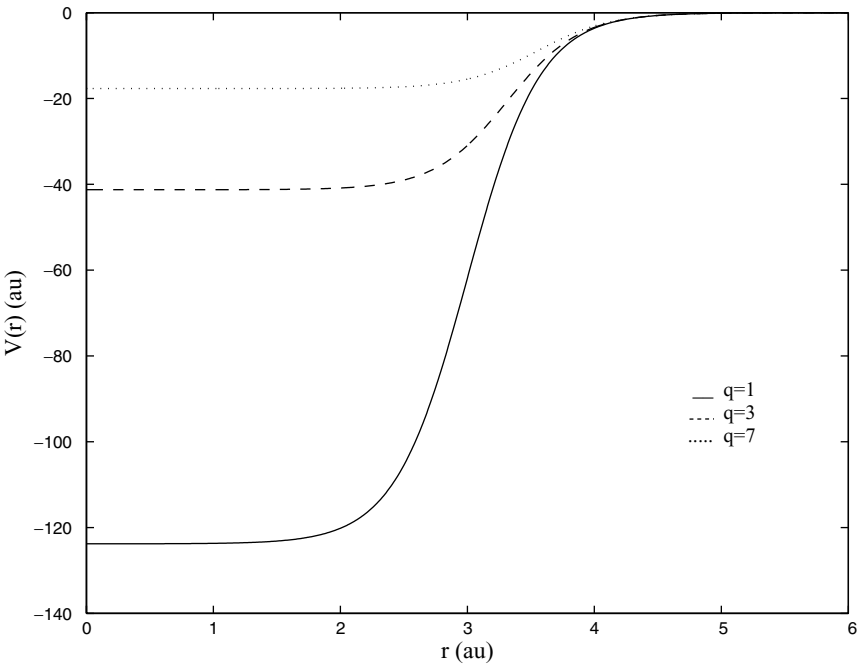

Fig. 1. General form of the generalized WS potential, i.e., $V_2 = 0$, for the s-states. The curves are drawn by using the appropriate optimized set of parameters for $R = 3.0$ au.

usual WS potential are drawn in Fig. 3 as function of the discrete level n using the three sets of parameters $R = 1.0, 2.0, 3.0$. Moreover, Figure 4 shows the lowest p-state ($n = 0$) energy shapes, in au, for the modified WS potential using the same sets of parameters in Fig. 1 together with $V_2 = V_1^2$ for the sake of simplicity.

We emphasize here that Fig. 3 (Fig. 4) have no lower bounds on the spectrum for $V_2 = 0$ ($V_2 = V_1^2$) cases, respectively. In this regard, it is found that the energy levels go into lower bound for $0 < n < 6$ ($0 < n < 10$) states and into higher bound for $n > 6$ ($n > 10$) states with an arbitrary choice of the physical parameter $R = 3.0$ ($R = 4.0$), respectively.

Finally, for $V_2 \neq 0$ case, the energy levels go into lower bound for $0 < n < 4$ states and higher bound for $n > 4$ states in $R = 3.0$ case. However, using the other sets of parameters $R = 1.0$ and $R = 2.0$, it goes into higher bound for $n > 1$.

Obviously, the modified WS potential stimulates different confinement barriers $R \geq 2$, very well and can be used in the analysis of the energy spectra of confined systems.

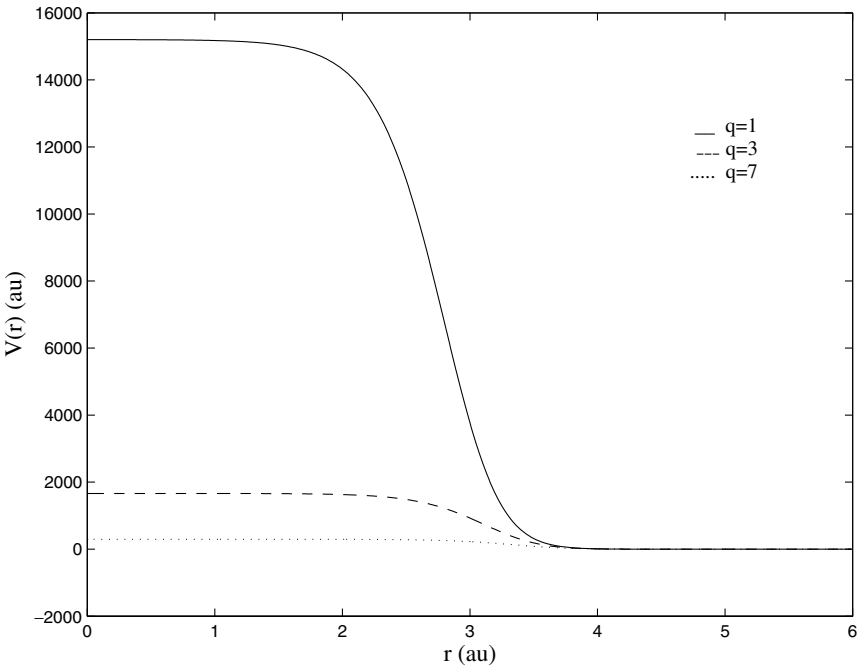

Fig. 2. General form of the modified WS potential for the lowest p-state. The curves are shown for $R = 3.0$ au with $V_2 = V_1^2$.

6. SUMMARY

In this work we have utilized NU method and solved the radial SE for the modified WS model potential with the angular momentum $l = 0$ ($V_2 = 0$) and $l \neq 0$ ($V_2 \neq 0$). A particularly interesting result of our investigation is that all the $\mathcal{PT}$-symmetric Hamiltonians with potential parameters remain all purely real have a real bound energies E_n with $n \geq 0$ for Hermitian case and real positive in contrary to expectation if one lets $\alpha \to i\alpha_I$ in the modified WS potential. Therefore, for non-Hermitian case, the spectrum is real for $V_2 = 0$ but complex conjugate for some values of $V_2 \neq 0$ in the modified WS potential. Further, when α and V_1 parameters are purely complex, it is seen that the number of discrete levels for bound states is given only by the real part of energy eigenvalues. Thus, for a $\mathcal{PT}$-symmetric Hamiltonians the exactness of $\mathcal{PT}$-symmetry implies the reality of spectrum. More specifically, if an eigenfunction $\psi_{nq}(s)$ is a $\mathcal{PT}$-invariant, $\mathcal{PT}$ $\psi_{nq}(s) = \psi_{nq}(s)$, then the corresponding eigenvalue of $E_{n,l}$ is real. The exact $\mathcal{PT}$-symmetry is a sufficient condition. But for a given $\mathcal{PT}$-symmetric Hamiltonian, it is not easy to determine the exactness of $\mathcal{PT}$-symmetry without actually solving

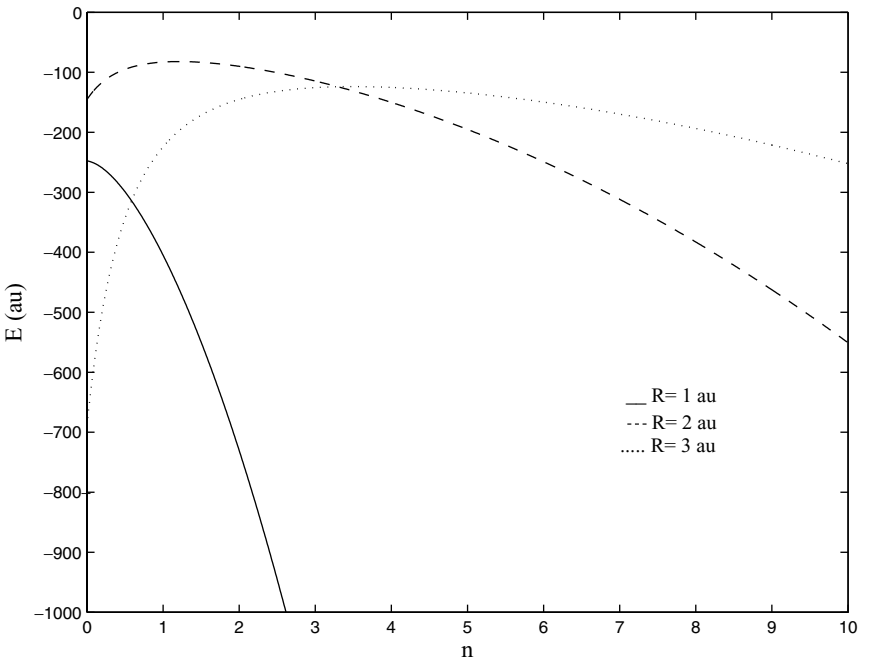

Fig. 3. Shape of the s-states energy eigenvalues for the usual WS potential, i.e., $q = 1$, with respect to the quantum number n for $R = 1.0$, 2.0 and 3.0 au.

the corresponding radial SE. In this regard, the wave functions are physical and energy eigenvalues are in good agreement with the results obtained by the other methods (Berkdemir *et al*., 2006b).

On the other hand, the effect of the centrifugal barrier potential which goes as $1/r^2$ was replaced by a term having exactly the generalized WS form but of a second degree in order to reproduce its effect. This $l \neq 0$ term has its physical basis arising from the superpotential partner of the generalized WS potential in SUSYQM (Berkdemir *et al*., 2006b). Hence, this new barrier term retakes the exact form of the original potential (Berkdemir *et al*., 2006a) but with a small perturbed strength factor; say V_2 (Berkdemir and Han, 2005; Berkdemir *et al*., 2006b,c). According to the complex quantum mechanics (Bender, Brody, and Jones, 2002), the eigenvalues of the conversion $\alpha \rightarrow i\alpha_I$ are not simultanously eigenstates of $\mathcal{PT}$-operator.

Finally, we point out that the exact results obtained for the proposed modified WS potential may have some interesting applications in the study of different quantum mechanical systems of nuclear physics such as in nuclear scattering.

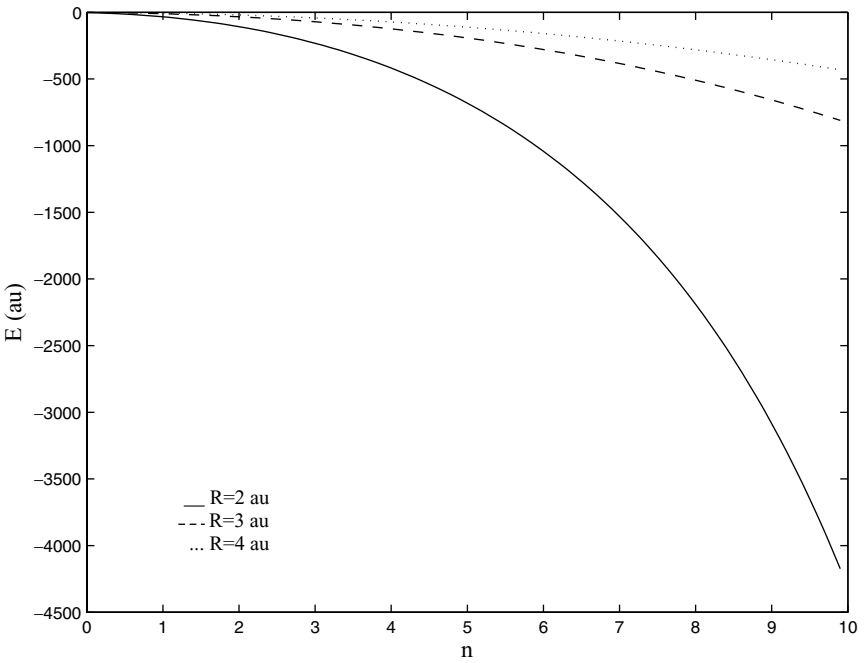

Fig. 4. Shape of the lowest p-state eenergy eigenvalues of the undeformed modified WS potential with respect to the quantum number n using the parameters for $R = 2.0$, 3.0 and 4.0 au together with $V_2 = V_1^2$.

ACKNOWLEDGMENTS

This research was partially supported by the Scientific and Technological Research Council of Turkey. The authors thank the referees and the editors for their helpful comments. We also thank Prof. C. Tezcan for his useful comments. Sameer M. Ikhdair acknowledges his family members for their love, encouragement and assistance.

REFERENCES

Ahmed, Z. (1991). *Physics Letters A* **157**, 1.
Ahmed, Z. (1993). *Physical Review A* **47**, 4757.
Ahmed, Z. (2001a). *Physics Letters A* **282**, 343.
Ahmed, Z. (2001b). *Physics Letters A* **290**, 19.
Aktaş, M. and Sever, R. (2004). *Journal of Molecular Structure* **710**, 219.
Arai, A. (1991). *J. Math. Anal. Appl.* **158**, 63.
Bagchi, B. and Quesne, C. (2000). *Physics Letters A* **273**, 285.
Barclay, D. T., *et al.* (2002). *Physics Letters A* **305**, 231.

Barton, G. (1986). *Annals of Physics* **166**, 322.
Bekmezci, A., Ikhdair, S. M., Magdy, M. A., and Sever, R. (1993). *Hadronic Journal* **16**, 339.
Bender, C. M., Boettcher, S., and Meisenger, P. N. (1999a). *Journal of Mathematical Physics* **40**, 2201.
Bender, C. M., Dunne, G. V., and Meisenger, P. N. (1999b). *Physics Letters A* **252**, 272.
Bender, C. M., Dunne, G. V., Meisenger, P. N., and Şimşek, M. (2001). *Physics Letters A* **281**, 311.
Bender, C. M., Brody, D. C., and Jones, H. F. (2002). *Physical Review Letters* **89**, 270402.
Bender, C. M. and Boettcher, S. (1998a). *Journal of Physics A* **31**, 1273.
Bender, C. M. and Boettcher, S. (1998b). *Physical Review Letters* **80**, 5243.
Berkdemir, C., Berkdemir, A., and Sever, R. (2005). *Physical Review C* **72**, 027001.
Berkdemir, A., Berkdemir, C., and Sever, R. (2006a). *Modern Physics Letters A* **21**, 2087.
Berkdemir, C., Berkdemir, A., and Sever, R. (2006b). *Nuclear Physics A* **770** 32.
Berkdemir, C., Berkdemir, A., and Han, J. (2006c). *Chemical Physics Letters* **417**, 326.
Berkdemir, C. and Han, J. (2005). *Chemical Physics Letters* **409**, 203.
Brack, M. (1993). *Rev. Mod. Phys*. **65**, 677.
Bulgac, A. and Lewenkopf, C. (1993). *Phys. Rev. Lett*. **71**, 4130.
Cannato, F., Junker, G., and Trost, J. (1998). *Physics Letters A* **246**, 219.
Chamon, L. C., *et al*. (1996). *Nuclear Physics A* **597**, 253.
Christensen, R. P. and Winther, A. (1976). *Physics Letters B* **65**, 19.
Costa, L. S., Prudenter, F. V., Acioli, P. H., Soares Neto, J. J., and Vianna, J. D. M. (1999). *Journal of Physics B* **32**, 2461.
Delabaere, E. and Trinh, D. T. (2000a). *Journal of Physics A* **33**, 8771.
Delabaere, E. and Trinh, D. T. (2000b). *Journal of Physics A* **33**, 8771.
Eğrifes, H., Demirhan, D., and Büyükkılıç, F. (1999a). *Physica Scripta* **59**, 90.
Eğrifes, H., Demirhan, D., and Büyükkılıç, F. (1999b). *Physica Scripta* **60**, 195.
Eğrifes, H. and Sever, R. (2005). *Physics Letters A* **344**, 117.
Fernández, F. M., Guardiola, R., Ros, J., and Znojil, M. (1998). *Journal of Physics A* **31**, 10105.
Filho, E. D. and Ricotta, R. M. (1995). *Modern Physics Letters A* **10**, 1613.
Flügge, S. (1971). *Practical Quantum Mechanics I*, Springer-Verlag, Berlin.
Hagino, K., *et al.* (2001). [arXiv:nucl-th/0110065] A talk given at the 4th Italy-Japan Symposium on Heavy-Ion Physics Tokyo, Japan.
Hamamato, I., Lukyanov, S., and Zhang, X. Z. (2001). *Nuclear Physics A* **683**, 255.
Ikhdair, S. M., *et al*. (1992). *Turkish Journal of Physics* **16**, 510.
Ikhdair, S. M., *et al*. (1993a) *Turkish Journal of Physics* **17**, 474.
Ikhdair, S. M., *et al*. (1993b). *Hadronic Journal* **16**, 57.
Ikhdair, S. M., *et al*. (1994a). *Hadronic Journal* **17**, 151.
Ikhdair, S. M., *et al*. (1994b). *Hadronic Journal Supplement* **19**, 417.
Ikhdair, S. and Sever, R. (1992a). *Hadronic Journal* **15**, 375.
Ikhdair, S. M. and Sever, R. (1992b). *Zeitschrift fur Physik C (Particles and Fields)* **56**, 155.
Ikhdair, S. M. and Sever, R. (1992c). *Hadronic Journal* **15**, 389.
Ikhdair, S. M. and Sever, R. (1993a). *Zeitschrift fur Physik C (Particles and Fields) C* **58**, 153.
Ikhdair, S. M. and Sever, R. (1993b). *Zeitschrift für Physik D* **28**, 1.
Ikhdair, S. M. and Sever, R. (2003). *International Journal of Modern Physics A* **18**, 4215.
Ikhdair, S. M. and Sever, R. (2004). *International Journal of Modern Physics A* **19**, 1771.
Ikhdair, S. M. and Sever, R. (2005a). *International Journal of Modern Physics A* **20**, 4035.
Ikhdair, S. M. and Sever, R. (2005b). [arXiv:quant-ph/0511209] to appear in *International Journal of Modern Physics A*.
Ikhdair, S. M. and Sever, R. (2005c). *International Journal of Modern Physics A* **20**, 6509.
Ikhdair, S. M. and Sever, R. (2005d). [arXiv:quant-ph/0508009] to appear in *Journal of Mathematical Chemistry*.

Ikhdair, S. M. and Sever, R. (2005e). [arXiv:hep-ph/0504107] to appear in *International Journal of Modern Physics A*.
Ikhdair, S. M. and Sever, R. (2006a). *International Journal of Modern Physics A* **21**, 2191.
Ikhdair, S. M. and Sever, R. (2006b). [arXiv:quant-ph/0604073] to appear in *Journal of Mathematical Chemistry*.
Ikhdair, S. M. and Sever, R. (2006c). *International Journal of Modern Physics A* **21**, 3989.
Ikhdair, S. M. and Sever, R. (2006d). [arXiv:quant-ph/0604078] to appear in *Journal of Mathematical Chemistry*.
Ikhdair, S. M. (2005). [arXiv:hep-ph/0504176] to appear in *International Journal of Modern Physics E*.
Jaminon, M., Jeukenne, J. P., and Mahaux, C. (1986). *Physical Review C* **34**, 468.
Japaridze, G. S. (2002). *Journal of Physics* **35**, 1709.
Jia, C. S., Zeng, X. L., and Sun, L. T. (2002). *Physics Letters A* **294**, 185.
Khare, A. and Mandal, B. P. (2000). *Physics Letters A* **272**, 53.
Khare, A. and Sukhatme, U. P. (1988). *Journal of Physics A* **21**, L501.
Kretschmer, R. and Szymanowski, L. (2001a). [arXiv:quant-ph/0105054].
Lévai, G. and Znojil, M. (2002). *Journal of Physics A* **35**, 8793.
Landau, L. D. and Lifshitz, E. M. (1958). *Quantum Mechanics*, Pergamon Press, London.
Macrobert, T. M. (1962). *Foundations of a Complex Variables*, London, Macmillan.
Magnus, W., Oberhettinger, F., and Soni, R. P. (1966). *Formulas and Theorems for the Special Function of Mathematical Physics*, 3rd edn. Berlin, Springer.
Mezincescu, G. A. (2000). *Journal of Physics A* **33**, 4911.
Mondal, C. K., Maji, K., and Bhattacharyya, S. P. (2001). *Physics Letters A* **291**, 203.
Morse, P. M. and Feshbach, H. (1953). *Methods of Theoretical Physics*, McGraw-Hill Book Company, Ltd. New York. pp. 1650–1660.
Mostafazadeh, A. (2002a). *Journal of Mathematical Physics* **43**, 205.
Mostafazadeh, A. (2002b). *Journal of Mathematical Physics* **43**, 2814.
Mostafazadeh, A. (2002c). *Journal of Mathematical Physics* **43**, 3944.
Nikiforov, A. F. and Uvarov, V. B. (1988). *Special Functions of Mathematical Physics*, Birkhauser, Basel.
Prudrinkov, A. P., Brychkov, Y. A., and Marichev, O. I. (1986). *Integrals and Series*, New York, Gordon and Breach.
Sahu, B., Agarwalla, S. K., and Shastry, C. S. (2002). *Journal of Physics A* **35**, 4349.
Sezgo, G. (1939). *Orthogonal Polynomials*, American Mathematical Society, New York.
Shin, K. C. (2001). *Journal of Mathematical Physics* **42**, 2513.
Silva, C. P., *et al*. (2001). *Nuclear Physics A* **679**, 287.
Şimşek, M. and Eğrifes, H. (2004). *Journal of Physics A (Mathematical and General)* **37**, 4379.
Taylor, B. N. (1984). *Physical Constants Table, Reviews of Modern Physics* **56**, 531.
Yasuk, F., Berkdemir, C., Berkdemir, A., and Önem, C. (2005). *Physica Scripta* **71**, 340.
Yeşiltaş, Ö., Şimşek, M., Sever, R., and Tezcan, C. (2003). *Physica Scripta* **67**, 472.
Znojil, M. and Tater, M. (2001). *Journal of Physics A* **34**, 1793.
Znojil, M. (1999). *Physics Letters A* **264**, 108.
Znojil, M. (2000). *Journal of Physics A* **33**, 4203.